SINC APPROXIMATION OF THE HEAT DISTRIBUTION ON THE BOUNDARY OF A TWO-DIMENSIONAL FINITE SLAB*

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Abstract : We consider the two-dimensional problem of recovering globally in time the heat distribution on the surface of a layer inside of a heat conducting body from two interior temperature measurements. The problem is ill-posed. The approximation function is represented by a two-dimensional Sinc series and the error estimate is given.

Key words and phrases : heat equation, heat distribution, Sinc series, ill-posed problem, regularization.

Mathematics Subjects Classification 2000. 35K05, 31A25, 44A35

1. Introduction

In this paper, we consider the problem of recovering the heat distribution on the surface of a thin layer inside of a heat conducting body from transient temperature measurements. The problem is raised in many applications in Physics and Geology. In fact, in many physical situation (see, e.g. [B]) we cannot attach a temperature sensor at the surface of the body (for example, the skin of a missile). On the other hand, we can easily measure the temperature history at an interior point of the body. Hence, to get the heating history in the body, we want to use temperature measured in the interior of the body. In the simplest model, the heat-conducting body is assumed to have a constant conductivity and represented by the half-line $x > 0$ (see, e.g. [C, EM, LN, TV]),. While giving many useful results, this model is not suitable for the case of a body having a series of superposed layers, each of which has a constant conductivity.

Precisely, we shall consider the problem corresponding to a thin layer of the body represented by the strip $\mathbf{R}\times(0,2)$, say. Let u be the temperature in the strip. For the uniqueness of solution, we shall have to measure the temperature history at two interior lines $\mathbf{R}\times\{y=1\}$ and $\mathbf{R}\times\{y=2\}$. From these measurements, we can identify uniquely the heating history inside of the layer (see, e.g., [B]). The problem is of finding the surface heat distribution $u(x,0,t) = v(x,t)$. In fact, despite uniqueness, the global solution in $L^2(\mathbf{R}\times\mathbf{R}_+)$ is unstability and hence, in this point of view, a sort of regularization is in order.

As discussed in the latter paragraph, the main purpose of our paper is to present a regularization of the problem. Moreover, an effective way of approximating the heat distribution v is also worthy of considering. There are many methods for regularizing the equation (see [TA, B]). In the most common scheme (see, e.g., [Blackwell]), the computation is divided into two steps. In the first step, one considers the problem of finding the heat flux $u_y(x,1,t)$ from the interior measurements $u(x,1,t), u(x,2,t)$. The problem is classical and can be changed to the one of finding the solution of a convolution of Volterra type which can be solved in any

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finite time interval by the iteration (see, e.g., [F]). But even in the "classical" problem as mentioned in many documents, it is worthy of insisting that the problem is *ill-posed* if we consider the problem over the whole time interval $\mathbf{R}_+$ with respect to the L^2-norm and the literature on this way is very scarce. In the second step, one considers the "really" ill- posed problem which is of recovering the surface temperature history $u(x, 0, t)$ from data $u(x, 1, t), u_y(x, 1, t)$. In the present paper, we can (and shall) regularize the function $u(x, 0, t)$ *directly* from $u(x, 1, t), u(x, 2, t)$ *without using* the flux function $u_y(x, 1, t)$. We emphasize that, using our method, we can unify two steps of the classical scheme and find simultaneously two functions $u(x, 0, t)$ and $u_y(x, 1, t)$. However, our main purpose is of regularizing the surface temperature. Hence, we shall omit the problem of finding the interior heat flux $u_y(x, 1, t)$.

Moreover, using the method of truncated integration, one can approximate the Fourier transform of the solution by a function having a *compact support* in $\mathbf{R}^2$. Therefore, the solution can be represented by an expansion of two dimensional Sinc series (see [AGTV]). The Sinc method is based on the Cardinal functions

$$S(p, d)(z) = \frac{sin[(\pi(z - pd))/d]}{\pi(z - pd)/d}, \;\; p \in \mathbf{Z}, d > 0$$

which dates back to the works of many mathematicians (Bohr, de la Vallee Poussin, E. T. Whittaker, ...). The one dimensional version of the method is studied very clearly and completely in [S]. Some primary results related to the two dimensional Sinc approximation were given in [AGTV]. As is known, the Sinc series converges very rapidly at an incredible $0(e^{-cn^{1/2}})$ rate, where $c > 0$ and n is the dimension of approximation (see [S]). Hence, this method, which is new in our knowledge, is very effective.

The remainder of the present paper is divided into two sections. In Section 2 , we state precisely the problem, change it into an integral equation of convolution type, and state main results of our paper. In Section 3, we give the proof of main results.

2. Notations and main results

Consider the problem of determining the heat distribution

$$u(x, 0, t) = v(x, t) \tag{1}$$

where u satisfies

$$\Delta u - \frac{\partial u}{\partial t} = 0 \quad x \in \mathbf{R},\; 0 < y < 2,\; t > 0, \tag{2}$$

subject to the boundary conditions

$$u(x, 2, t) = g(x, t), \;\; x \in \mathbf{R}, \;\; t > 0, \tag{3}$$

$$u(x, 1, t) = f(x, t), \;\; x \in \mathbf{R}, \;\; t > 0, \tag{4}$$

and the initial condition

$$u(x, y, 0) = 0, \;\; x \in \mathbf{R}, \;\; 0 < y < 2. \tag{5}$$

Here f, g are given. We shall transform the problem (1)-(5) into a convolution intergral equation.

Put

$$\Gamma(x, y, t, \xi, \eta, \tau) = \frac{1}{4\pi(t - \tau)} \exp\left(-\frac{(x - \xi)^2 + (y - \eta)^2}{4(t - \tau)}\right) \tag{6}$$

and

$$G(x,y,t,\xi,\eta,\tau)=\Gamma(x,y,t,\xi,\eta,\tau)-\Gamma(x,4-y,t,\xi,\eta,\tau). \tag{7}$$

We have

$$G_{\xi\xi}+G_{\eta\eta}+G_\tau=0.$$

Integrating the identity

$$div(G\nabla u-u\nabla G)-\frac{\partial}{\partial\tau}(uG)=0$$

over the domain $\mathbf{R}\times(1,2)\times(0,t-\varepsilon)$ and letting $\varepsilon\to 0$, we have

$$\begin{aligned}\int\limits_{-\infty}^{+\infty}\int\limits_0^t g(\xi,\tau)G_\eta(x,y,t,\xi,2,\tau)d\xi d\tau \quad &+\quad \int\limits_{-\infty}^{+\infty}\int\limits_0^t G(x,y,t,\xi,1,\tau)u_y(\xi,1,\tau)d\xi d\tau\\ &-\quad \int\limits_{-\infty}^{+\infty}\int\limits_0^t f(\xi,\tau)G_\eta(x,y,t,\xi,1,\tau)d\xi d\tau+u(x,y,t)=0.\end{aligned}$$

Hence

$$\begin{aligned}&\int\limits_{-\infty}^{+\infty}\int\limits_0^t G(x,y,t,\xi,1,\tau)u_y(\xi,1,\tau)d\xi d\tau=-u(x,y,t)+\\ &\int\limits_{-\infty}^{+\infty}\int\limits_0^t G_\eta(x,y,t,\xi,1,\tau)f(\xi,\tau)d\xi d\tau-\int\limits_{-\infty}^{+\infty}\int\limits_0^t g(\xi,\tau)G_\eta(x,y,t,\xi,2,\tau)d\xi d\tau.\end{aligned} \tag{8}$$

Letting $y\to 1^+$ in (8), we have

$$\begin{aligned}&\int\limits_{-\infty}^{+\infty}\int\limits_0^t \left[\frac{1}{2\pi(t-\tau)}\exp\left(-\frac{(x-\xi)^2}{4(t-\tau)}\right)-\frac{1}{2\pi(t-\tau)}\exp\left(-\frac{(x-\xi)^2+4}{4(t-\tau)}\right)\right]u_y(\xi,1,\tau)d\xi d\tau\\ &=-f(x,t)-\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\int\limits_0^t \frac{1}{(t-\tau)^2}\exp\left(-\frac{(x-\xi)^2+4}{4(t-\tau)}\right)f(\xi,\tau)d\xi d\tau\\ &+\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\int\limits_0^t g(\xi,\tau)\frac{1}{(t-\tau)^2}\exp\left(-\frac{(x-\xi)^2+1}{4(t-\tau)}\right)d\xi d\tau.\end{aligned} \tag{9}$$

We put $N(x,y,t,\xi,\eta,\tau)=\Gamma(x,y,t,\xi,\eta,\tau)-\Gamma(x,-y,t,\xi,\eta,\tau)$

Integrating the identity

$$div(N\nabla u-u\nabla N)-\frac{\partial}{\partial\tau}(uN)=0$$

over the domain $(-n,n)\times(0,1)\times(0,t-\varepsilon)$ and letting $n\to\infty,\varepsilon\to 0$

$$\begin{aligned}
\int_{-\infty}^{+\infty}\int_0^t N(x,y,t,\xi,1,\tau)u_y(\xi,1,\tau)d\xi d\tau \quad &- \quad \int_{-\infty}^{+\infty}\int_0^t f(\xi,\tau)N_\eta(x,y,t,\xi,1,\tau)d\xi d\tau \\
+\int_{-\infty}^{+\infty}\int_0^t v(\xi,\tau)N_\eta(x,y,t,\xi,0,\tau)d\xi d\tau \quad &- \quad u(x,y,t)=0. \qquad (10)
\end{aligned}$$

Letting $y\to 1^-$, the identity (10) becomes

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_0^t \frac{1}{t-\tau}\left[\exp\left(-\frac{(x-\xi)^2}{4(t-\tau)}\right)-\exp\left(-\frac{(x-\xi)^2+4}{4(t-\tau)}\right)\right]u_y(\xi,1,\tau)d\xi d\tau \\
&-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_0^t f(\xi,\tau)\frac{1}{(t-\tau)^2}\exp\left(-\frac{(x-\xi)^2+4}{4(t-\tau)}\right)d\xi d\tau \\
&+\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_0^t \frac{1}{(t-\tau)^2}\exp\left(-\frac{(x-\xi)^2+1}{4(t-\tau)}\right)v(\xi,\tau)d\xi d\tau-3f(x,t)=0 \qquad (11)
\end{aligned}$$

From (9) and (11), we have the main convolution equation

$$\begin{aligned}
&-\frac{1}{\pi}\int_{-\infty}^{+\infty}\int_0^t \frac{1}{(t-\tau)^2}\exp\left(-\frac{(x-\xi)^2+4}{4(t-\tau)}\right)f(\xi,\tau)d\xi d\tau \\
&+\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_0^t g(\xi,\tau)\frac{1}{(t-\tau)^2}\exp\left(-\frac{(x-\xi)^2+1}{4(t-\tau)}\right)d\xi d\tau \\
&+\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_0^t \frac{1}{(t-\tau)^2}\exp\left(-\frac{(x-\xi)^2+1}{4(t-\tau)}\right)v(\xi,\tau)d\xi d\tau-4f(x,t)=0
\end{aligned}$$

which can be rewritten as

$$S*v(x,t)=2R*f(x,t)-S*g(x,t)+4f(x,t) \qquad (12)$$

where we define that $v(x,t)=f(x,t)=g(x,t)=0$ as $t<0$,

$$R(x,t)=\begin{cases}\frac{1}{t^2}\exp\left(-\frac{x^2+4}{4t}\right) & (x,t)\in\mathbf{R}\times[0,+\infty)\\ 0 & (x,t)\in\mathbf{R}\times(-\infty,0)\end{cases}$$

and

$$S(x,t)=\begin{cases}\frac{1}{t^2}\exp\left(-\frac{x^2+1}{4t}\right) & (x,t)\in\mathbf{R}\times[0,+\infty)\\ 0 & (x,t)\in\mathbf{R}\times(-\infty,0)\end{cases}$$

Put $F(x,t)=2R*f(x,t)-S*g(x,t)+4f(x,t)$.

Taking the Fourier-transform of (12), we get

$$\hat{S}(z,r)\hat{v}(z,r)=\hat{F}(z,r)$$

where

$$\hat{S}(z,r) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} S(z,r)e^{-i(xz+tr)}dxdt$$
$$= 2e^{-\frac{1}{\sqrt{2}}\sqrt{\sqrt{z^4+r^2}+z^2}}\left[\cos\frac{1}{\sqrt{2}}\sqrt{\sqrt{z^4+r^2}-z^2} - isgn(r)\sin\frac{1}{\sqrt{2}}\sqrt{\sqrt{z^4+r^2}-z^2}\right]$$

and

$$\left|\hat{S}(z,r)\right| = 2e^{-\frac{1}{\sqrt{2}}\sqrt{\sqrt{z^4+r^2}+z^2}}.$$

We have

Theorem 1

Let $\gamma \in (0,2)$ and $\varepsilon \in \left(0, e^{\frac{-3}{\gamma}}\right)$.

Assume that $v_0 \in L^2(\mathbf{R}^2)$ is the (unique) solution of (12) corresponding to the exact data $f_0, g_0 \in L^2(\mathbf{R}^2)$ and that $f, g \in L^2(\mathbf{R}^2)$ are measured data satisfying $\|f - f_0\|_2 \le \varepsilon$ and $\|g - g_0\|_2 \le \varepsilon$ where $\|.\|_2$ is the $L^2(\mathbf{R}^2)$-norm.

Then we can construct from g, f a function $v_\varepsilon \in L^2$ such that

$$\|v_\varepsilon - v_0\|_2 \le \sqrt{C\varepsilon^{2-\gamma} + \eta(\varepsilon)}$$

where C is constant and $\eta(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$.

Moreover, if we assume in addition that $v_0 \in H^m(\mathbf{R}^2) \cap L^1(\mathbf{R}^2), m > 0$ and $0 < \varepsilon < min\{e^{-e^2}, e^{-4m^2}\}$ then

$$\|v_\varepsilon - v_0\|_2 < D\left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{-m}$$

where $D > 0$ depends on v_0.

Theorem 2

With v_ε as in theorem 1, we have

$$v_\varepsilon(x,t) = \sum_{n=-\infty}^{+\infty}\sum_{|m|\le|n|} v_\varepsilon\left(\frac{m\pi}{a_\varepsilon}, \frac{n\pi}{a_\varepsilon}\right)S(m, \pi/a_\varepsilon)(x)S(n, \pi/a_\varepsilon)(t)$$

where

$$S(p,d)(z) = \frac{sin[\pi(z-pd)/d]}{\pi(z-pd)/d}, \ \ p \in \mathbf{Z}, d > 0.$$

3. Proofs

Proof of theorem 1

We put

$$F(x,t) = 2R * f(x,t) - S * g(x,t) + 4f(x,t)$$

and

$$F_0(x,t) = 2R * f_0(x,t) - S * g_0(x,t) + 4f_0(x,t)$$

then

$$\begin{aligned}\left\|\hat{F}-\hat{F}_0\right\|_2 &= \|F-F_0\|_2 \\ &\le (4+2\|R\|_1)\|f-f_0\|_2+\|S\|_1\|g-g_0\|_2 \\ &\le (4+2\|R\|_1+\|S\|_1)\,\varepsilon.\end{aligned}$$

Put

$$v_\varepsilon(x,t)=\frac{1}{2\pi}\int\limits_{D_\varepsilon}\frac{\hat{F}(z,r)}{\hat{S}(z,r)}e^{i(xz+tr)}dzdr \tag{13}$$

where $D_\varepsilon=\left\{(z,r)/\ |z|\le b_\varepsilon \text{ and } |r|\le b_\varepsilon^2\right\}$ with $b_\varepsilon=\frac{1}{\sqrt{2}\sqrt{\sqrt{2}+1}}\ln\frac{4}{\varepsilon^\gamma}$.

We have

$$\begin{aligned}\|v_\varepsilon-v_0\|_2^2=\|\hat{v}_\varepsilon-\hat{v}_0\|_2^2 &= \int\limits_{D_\varepsilon}\left|\frac{\hat{F}(z,r)-\hat{F}_0(z,r)}{\hat{S}(z,r)}\right|^2dzdr+\int\limits_{\mathbf{R}^2\setminus D_\varepsilon}|\hat{v}_0(z,r)|^2\,dzdr \\ &\le \varepsilon^{2-\gamma}\left(4+2\|R\|_1+\|S\|_1\right)^2+\int\limits_{\mathbf{R}^2\setminus D_\varepsilon}|\hat{v}_0(z,r)|^2\,dzdr\end{aligned}$$

If we put $\eta(\varepsilon)=\int\limits_{\mathbf{R}^2\setminus D_\varepsilon}|\hat{v}_0(z,r)|^2\,dzdr$ then $\eta(\varepsilon)\to 0$ as $\varepsilon\downarrow 0$.

Now, we assume $v_0\in H^m(\mathbf{R}^2), m>0$, put

$$a_\varepsilon=\frac{\sqrt{2}}{\sqrt{\sqrt{2}+1}}\ln\left(\frac{1/\varepsilon}{\ln^m(1/\varepsilon)}\right)>1$$
$$Q_\varepsilon=[-a_\varepsilon,a_\varepsilon]\times[-a_\varepsilon,a_\varepsilon]$$

and

$$v_\varepsilon(x,t)=\frac{1}{2\pi}\int\limits_{Q_\varepsilon}\frac{\hat{F}(z,r)}{\hat{S}(z,r)}e^{i(xz+tr)}dzdr$$

We have

$$\begin{aligned}\|v_\varepsilon-v_0\|_2^2 &= \int\limits_{Q_\varepsilon}\frac{\left|\hat{F}(z,r)-\hat{F}_0(z,r)\right|^2}{\left|\hat{S}(z,r)\right|^2}dzdr+\int\limits_{\mathbf{R}^2\setminus Q_\varepsilon}|\hat{v}_0(z,r)|^2\,dzdr \\ &\le 4\varepsilon^2(4+2\|R\|_1+\|S\|_1)^2e^{\sqrt{2}\sqrt{\sqrt{2}+1}a_\varepsilon}+\int\limits_{\mathbf{R}^2\setminus Q_\varepsilon}\frac{(z^2+r^2)^m\,|\hat{v}_0(z,r)|^2}{(z^2+r^2)^m}dzdr \\ &\le C_1\left(\varepsilon^2e^{\sqrt{2}\sqrt{\sqrt{2}+1}a_\varepsilon}+\frac{1}{(2a_\varepsilon^2)^m}\right)\end{aligned}$$

where

$$C_1=\max\left\{4\left(4+2\|R\|_1+\|S\|_1\right)^2,\left\|(z^2+r^2)^{m/2}\hat{v}_0(z,r)\right\|_2^2 2^m\right\}$$

This implies that

$$\begin{aligned}
\|v_\varepsilon - v_0\|_2^2 &\leq C_1 \left[\varepsilon^2 \left(\frac{1/\varepsilon}{\ln^m(1/\varepsilon)} \right)^2 + \frac{1}{\ln^{2m}\left(\frac{1/\varepsilon}{\ln^m(1/\varepsilon)} \right)} \right] \\
&\leq C_1 \left[\frac{1}{\ln^{2m}(1/\varepsilon)} + \frac{1}{\ln^{2m}\left(\frac{1/\varepsilon}{\ln^m(1/\varepsilon)} \right)} \right] \\
&\leq C_1 \left[\frac{1}{\ln^{2m}(1/\varepsilon)} + \frac{2^m}{\ln^{2m}(1/\varepsilon)} \right] = D^2 \frac{1}{\ln^{2m}(1/\varepsilon)}
\end{aligned}$$

where $D = \sqrt{C_1(1+2^m)}$.

This completes the proof.

Proof of theorem 2

We have

$$supp\ \hat{v_\varepsilon} \subset D_\varepsilon \subset [-a_\varepsilon, a_\varepsilon] \times [-a_\varepsilon, a_\varepsilon].$$

As in [AGLT], p.121, we have theorem 2.

This completes the proof.

4. Numerical results

We present some results of numerical comparison of the regularized representation of the solution given by theorem 2 and the corresponding exact solution of the problem.

Let the problem

$$\Delta u - \frac{\partial u}{\partial t} = 0, \ \ (x,y) \in \mathbf{R} \times (0,2), \ t > 0 \tag{14}$$

$$u(x,1,t) = \frac{1}{t} \mathrm{e}^{\frac{-x^2-1}{4t}}; \ \ u(x,2,t) = \frac{1}{t} \mathrm{e}^{\frac{-x^2-4}{4t}}; \ u(x,y,0) = 0 \tag{15}$$

whose the unknown is

$$v(x,t) = u(x,0,t), \quad x \geq x_0 > 0, \quad t > 0 \tag{16}$$

The exact solution of this problem is

$$v(x,t) = \frac{1}{t} \mathrm{e}^{\frac{-x^2}{4t}}.$$

The approximated solution is calculated from the expansion of two-dimensional Sinc series given by theorem 2 associated to formula (13) in which

$$\hat{F} = 4 \frac{\mathrm{e}^{-\sqrt{r^2+z^4}}}{\sqrt{r^2+z^4}}, \qquad \hat{S} = 2\mathrm{e}^{-\sqrt{r^2+z^4}}$$

Thus we have

$$\left\| \hat{F} - \hat{F}_0 \right\|_{L^2(\mathbf{R}^2)} = \varepsilon$$

which is a perturbation similar to the one operated on the two given functions f and g.

With $\varepsilon = \dfrac{1}{50}$, $N = 50$ (the size of the double series) and for $(x,t) \in [0.25, 1.3] \times [0,4]$ we have drawn the corresponding approximate surface solution $(x,t) \longrightarrow v_\varepsilon(x,t)$ in Fig.1.

To calculate the double integral in (10) we have used the rectangle rule which gives good accuracy if one integrates on the interval $[\varepsilon, 1/\varepsilon] \times [\varepsilon, 1/\varepsilon]$. The time of calculus with a good computer is very long: 2 hours for 900 points $M = (x,t) \in [0.25, 1.3] \times [0,4]$. It is the reason for which we are limited ourselves to a relatively small size of the double series ($N = 20$). For comparison in Fig.2 we have drawn the exact solution $(x,t) \longrightarrow v(x,t)$.

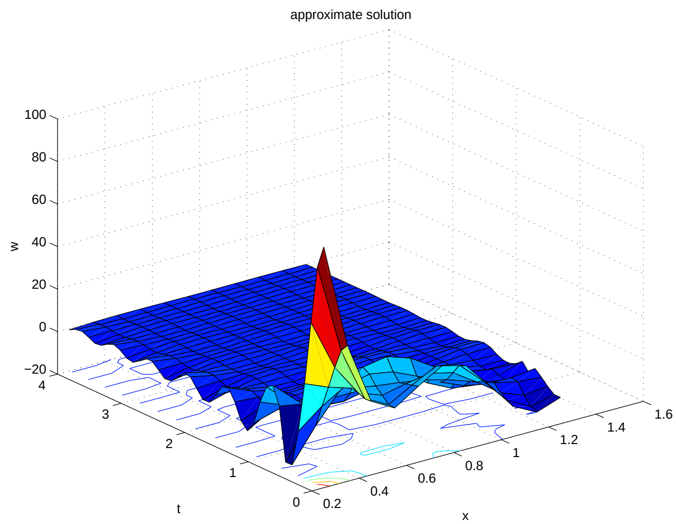

Fig.1: regularized solution of the problem (14), (15)

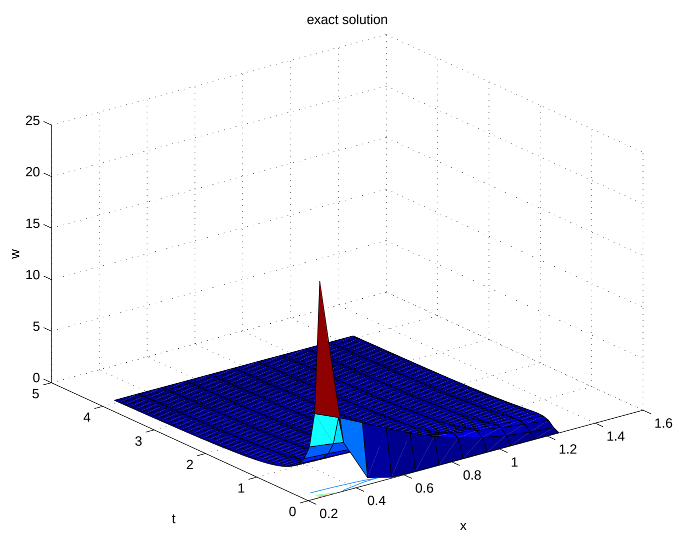

Fig.1: exact solution of the problem (14), (15)

Using the same method as previously we have drawn in Fig.3 the surface $(x,t) \in [0,1] \times [0,4] \longrightarrow v_\varepsilon(x,t)$ which is the regularization of the following problem

$$\Delta u - \frac{\partial u}{\partial t} = 0, \quad (x,y) \in \mathbf{R} \times (0,2), \; t > 0 \tag{17}$$

$$u(x,1,t)=0;\;\; u(x,2,t)=\frac{1}{t}\mathrm{e}^{\frac{-x^2-4}{4t}};\; u(x,y,0)=0 \tag{18}$$

the unknown being $v(x,t)=u(x,0,t)$. The exact solution

$$v(x,t)=-\frac{1}{t}\mathrm{e}^{\frac{-x^2-4}{4t}}.$$

is represented in Fig.4.

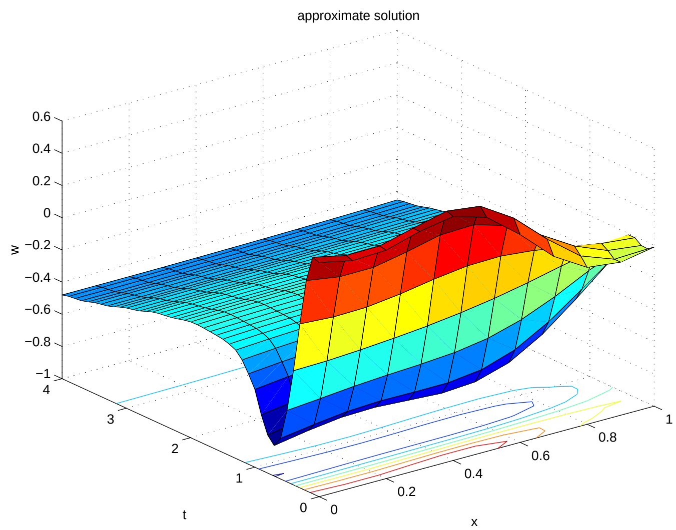

Fig.3: regularized solution of the problem (17), (18)

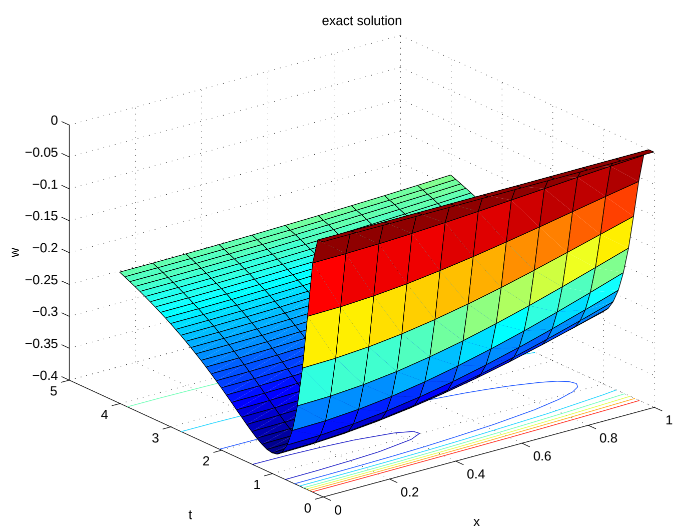

Fig.4: exact solution of the problem (17), (18)

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